LENGTH- MINIMIZERS

on the way to Pontryagin extremals

Outline. 1. Recall · sub-Riem distance

2. Existence of length-minimizers

3. Towards Pontryagin extremals

1. RECALL

DEF: M sub-Riem mfd, $q_0, q_1 \in M$, $\gamma : [0,T] \to M$ adm.

$$\ell(\gamma) := \int_0^T \underbrace{\|\dot\gamma(t)\|}_{\text{sub-Riem. norm}} \, dt = \int_0^T \underbrace{|u^*(t)|}_{\text{min control}} \, dt \qquad \text{length}$$

$$d(q_0, q_1) := \inf \{ \ell(\gamma) : \gamma \text{ adm.}, \gamma(0) = q_0, \gamma(T) = q_1 \} \qquad \text{distance}$$

Thm (Rashevskii-Chow): M sub-Riem mfd

Then: ·) (M, d) metric space

·) d induces the same topology on M

In part.: $d \cdot M \times M \longrightarrow \mathbb{R}$ is continuous.

Rmk: $d(q_0, q_1) < +\infty \qquad \forall q_0, q_1 \in M$ M connected

COR: (M, d) is locally compact.

$1\frac{1}{2}$. Exkursion: Free Sub-Riem. structures

$$U := M \times \mathbb{R}^m$$
$$\ni (q, u)$$

$\{e_1, \dots, e_m\}$: orth. basis of $\mathbb{R}^m$

Define $\quad f_i(q) := f(q, e_i) \qquad \forall i = 1, \dots, m$

$$\Longrightarrow \quad f(q,u) = f(q, \sum_i u_i e_i) = \sum_i u_i f_i(q) \quad , \; q \in M$$

__FACT:__ Every sub-Riem. str. (U, f) on M is equiv. to $(\bar{U}, \bar{f})$, where $\bar{U}$ trivial bundle.

equiv. as distributions [Def. 3.18]

$\rightsquigarrow$ In the following assume: $\exists$ global ~~generating family~~,
i.e. $\quad f_1, \dots, f_m$ globally def. on M
s.t. $\quad \dot{\gamma}(t) = \sum_{i=1}^m u_i(t) f_i(\gamma(t))$.

$$\Rightarrow \quad \ell(\gamma) = \int_0^T \left(\sum_i u_i^*(t)^2 \right)^{1/2} dt$$

min. control assoc. to γ

Optimal Control Problem:

$$\begin{cases} \dot{\gamma}(t) = \sum_i u_i(t) f_i(\gamma(t)) \\ \text{minimize} \int_0^T |u(t)| \, dt \\ \gamma(0) = q_0 \;, \; \gamma(T) = q_1 \qquad q_0, q_1 \in M \end{cases}$$

2. Existence of length-min.

DEF: γ adm. curve

γ is a length-min. if $\ell(\gamma) = d(\gamma(0), \gamma(T))$.

Rmk: ·) ex. of length-min. NOT guaranteed in general

·) length-min. are not unique

THM (semi-continuity): γ_n seq. of adm. curves par. by arclength s.t. $\gamma_n \longrightarrow \gamma$ uniformly on $[0,T]$ w/ $\liminf_n \ell(\gamma_n) < +\infty$.

Then: γ adm. and $\ell(\gamma) \leq \liminf_n \ell(\gamma_n)$.

COR: γ_n seq. of length-min. $\Rightarrow$ γ length-min.

Pf:

$$\ell(\gamma) \leq \liminf_n \ell(\gamma_n) = \liminf d(\gamma_n(0), \gamma_n(T)) = d(\gamma(0), \gamma(T)).$$

☒

Notation: $B_q(r) := \{q' \in M \mid d(q, q') < r\}$ open sub-Riem. ball

Thm. (Ex. length-min.): M sub-Riem. mfd., $q_0 \in M$

Assume $\overline{B_{q_0}(r)}$ is cpt. for some $r > 0$.

Then: For all $q_1 \in B_{q_0}(r)$ we have

$$d(q_0, q_1) = \min\{\ell(\gamma) \mid \gamma \text{ adm.}, \gamma(0) = q_0, \gamma(T) = q_1\}.$$

Pf: Fix $q_1 \in B_{q_0}(r)$,

$\gamma_n : [0,1] \longrightarrow M$ minimizing seq. of adm. traj. w/ cst. speed s.t. $\ell(\gamma_n) \longrightarrow d(q_0, q_1)$.

$\cdot)\ \ell(\gamma_n) \leq r \qquad \forall n \geq n_0$ large enough

$\overset{\Uparrow}{d(q_0,q_1) < r}$

WLOG: $\mathrm{image}(\gamma_n) \subset \underbrace{\overline{B}_{q_0}(r)}_{=:K} \qquad \forall n$

For all $n \geq n_0$: $\qquad \tau, t \in [0,1]$

$$|\gamma_n(t) - \gamma_n(\tau)| \leq \int_\tau^t |\dot{\gamma}_n(s)|\, ds$$

$$\overset{\text{free sub-Riem.}}{\leq} \int_\tau^t \sum_{i=1}^m |u_i(t)\, f_i(\gamma(t))|\, dt$$

$$\leq \int_\tau^t \left(\sum_{i=0}^m |f_i(\gamma(t))|^2\right)^{1/2} \cdot \left(\sum_{i=0}^m u_i(t)^2\right)^{1/2} dt$$

Define $C_K := \max\limits_{x \in K} \left(\sum_{i=1}^m |f_i(x)|^2\right)^{1/2}$.

$$\leq C_K \cdot |t-\tau| \cdot \underbrace{\ell(\gamma)}_{\leq r}$$

$\Rightarrow \gamma_n$ equicont. and unif. bdd

Ascoli-Arzelà: $\exists$ subseq. γ_n (still den. γ_n)
and Lipschitz curve $\gamma : [0,1] \to M$
s.t. $\gamma_n \to \gamma$ unif.

$\ell(\gamma) \leq d(q_0,q_1)$. $\qquad \Rightarrow \ell(\gamma) = d(q_0,q_1)$

$\blacksquare$

Cor: $q_0 \in M$

There ex. $\varepsilon > 0$ s.t. $\forall q_1 \in B_{q_0}(\varepsilon)$ there ex. a min. curve joining q_0 and q_1.

Prop: M sub-Riem.

TFAE: 1.) (M, d) complete

sub-Riem. Hopf-Rinow

2.) $\overline{B}(x, r)$ cpt. for every $x \in M$, $r > 0$

3.) $\exists\ \varepsilon > 0$ s.t. $\overline{B}(x, \varepsilon)$ is cpt. $\forall x \in M$

COR: (M, d) complete sub-Riem.

For every $q_0, q_1 \in M$ there ex. a length-minimizer γ joining q_0, q_1, i.e. $\ell(\gamma) = d(q_0, q_1)$.

3. Towards Pontryagin extremals

⚠ cannot use initial velocity to parametrize length-min. traj.

Reason:

If $\operatorname{rk}(q_0) < \dim M$, then $\dot\gamma(0)$ of γ starting at q_0 belong to the proper subspace $D_{q_0} \subset T_{q_0}M$

$\operatorname{rk}(q_0) = \dim D_{q_0}$

$D_{q_0} = f(U_{q_0}) \subset T_{q_0}M$

$\rightarrow \dim\{\text{adm. velocities}\} < \dim M$

Rashevskii–Chow & Ex. of length-min:

length-minimizers starting at q_0 cover a full nhd. of q_0 #

Right approach: use initial point and initial covector $\lambda_0 \in T^*_{q_0}M$ to param. length-min. traj.

think of a linear form annihilating the front

of a linear form
annihilating the front

THM. (Characterize Pontryagin extremals):

$\gamma : [0,T] \to M$ adm. curve that is length-min. param. by cst. speed

$\bar{u}$: corr. min. control

$P_{0,t}$: flow of the non-auton. vfield $f_{\bar{u}(t)} = \sum_{i=1}^{k} \bar{u}_i(t) \, f_i$

def. for $t \in [0,T]$
and x in a nhd. of $\gamma(0)$

Then: $\exists$ $\lambda_0 \in T^*_{\gamma(0)} M$ s.t., defining $\lambda(t) := (P_{0,t}^{-1})^* \lambda_0 \in T^*_{\gamma(t)} M$

we have ONE of the following:

(N) $\quad \bar{u}_i(t) \equiv \langle \lambda(t), f_i(\gamma(t)) \rangle \quad \forall i = 1, \dots, m$ $\leftarrow$ normal extremals

(A) $\quad 0 \equiv \langle \lambda(t), f_i(\gamma(t)) \rangle \quad \forall i = 1, \dots, m$ $\leftarrow$ abnormal extremals

$\hookrightarrow$ in (A): $\lambda_0 \neq 0$.

Rmk: ·) (N) $\Longrightarrow$ $\bar{u}$ smooth

·) Given $\lambda(t)$: (N) and (A) are mutually exclusive
unless $\bar{u}(t) = 0$ for a.e. $t \in [0,T]$

·) If $\mathcal{D}_{q_0} \neq T_{q_0} M$,

then trivial traj. is (N) and (A).

corr. to $\bar{u}(t) \equiv 0$

w/ assoc. $\lambda_0 = 0$

w/ assoc. $\lambda_0 \in \mathcal{D}_{q_0}^{\perp}$

·) A nontrivial traj. can also be both.

·) Riem. case: NO abnormal extremals

Q: How to find an explicit expression of extremals
for given sub-Riem. str.?

A: One may view Pontryagin extremals as

A: One may view Pontryagin extremals as sol. of a Ham. system

Ingredients of the Proof: (of Pontryagin extremal thm.)

Energy functional: $\gamma : [0,T] \to M$ adm. curve

$$E(\gamma) := \frac{1}{2} \int_0^T \| \dot{\gamma}(t) \|^2 \, dt$$

Rmk: ·) E is NOT invariant under reparametrisation

Lemma: Fix $T > 0$.

Ω_{q_0,q_1} : set of adm. curves, γ adm.

Then:

γ minimizer of E $\Leftrightarrow$ γ length-min. on Ω_{q_0,q_1} w/ cst. speed.

#

Outlook to the talk given by Max on 22/07/20:

THANKS, MAX :)

→ rewrite P.E. thm. geometrically

→ finally: small pieces of normal extremals are length-min.

Ham. encodes all info on sub-Riem. str.

Summary:

·) Optimal control problem

·) ex. of length-min.

·) characterization of Pontryagin extremals